Simple criterion for the occurrence of Bose-Einstein condensation and the Meissner-Ochsenfeld effect

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Abstract

We examine the occurrence of Bose-Einstein condensation in both nonrelativistic and relativistic systems with no self-interactions in a general setting. A simple condition for the occurrence of Bose-Einstein condensation is given. We show that condensation can occur only if $q \geq 3$, where q is the dimension associated with the continuous part of the eigenvalue spectrum of the Hamiltonian for nonrelativistic systems or the spatial part of the Klein-Gordon operator for relativistic systems. Furthermore we show that the criterion for the appearance of the Meissner-Ochsenfeld effect is closely connected with that for the appearance of Bose-Einstein condensation.

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I. INTRODUCTION

One of the most interesting properties of a system of bosons is that under certain conditions it is possible to have a phase transition at a critical value of the temperature in which a macroscopic fraction of the bosons can condense into the ground state. This was first predicted over 70 years ago for the ideal nonrelativistic Bose gas [1,2] and is nowadays well known to happen if the spatial dimension $D \geq 3$. (See [3] for the case $D = 3$ and [4] for general D.) The analogous phenomenon for ideal relativistic gases has also been studied and the criterion on the spatial dimension has been found to be the same as that for nonrelativistic gases [5–7].

Of course it is of great interest to see what influence different types of interactions have on the condensation. In nonrelativistic theories in certain models it is known what happens if interactions are included [8]. In relativistic theories a detailed study of $\lambda\phi^4$ theory at finite temperature and density has been given in flat [9,10] as well as in curved spacetime [11].

Rather than examine self-interacting fields, or the interactions among different quantum fields, a simpler problem is to study what happens for a quantum field under external conditions, where by external conditions we mean for example gravitational or electromagnetic background fields or simply boundary conditions imposed on the field. Several different situations have already been considered. Let us only mention the analysis in the presence of boundaries and static gravitational fields [12–20] and in the presence of (mainly constant) magnetic fields [21–32].

One of the main questions in all these kinds of considerations is, whether or not Bose-Einstein condensation can occur. In the above literature rather involved analysis has been done in order to arrive at the conclusion whether or not Bose-Einstein condensation can occur. Recently, the authors gave a very simple criterion to decide whether a system might condense or not [33]. Here, we plan to give more details of the calculations and to apply the criterion to a large class of examples. This will include the examples treated in the above mentioned literature as well as some new cases. Furthermore we will show how the criterion

might be used to decide if a system reveals the Meissner-Ochsenfeld effect or not.

It is worth emphasizing precisely what is meant by BEC since different definitions are possible. For the free Bose gas in three or more spatial dimensions the specific heat has a non-smooth behaviour at a critical temperature which signals a phase transition. In terms of the quantum field theory the phase transition may be interpreted as symmetry breaking with the scalar field developing a non-zero expectation value at the critical temperature. Associated with this phase transition is a sudden growth in the occupancy of the ground state. In the present paper we will adopt this as our definition of BEC. It is possible to have a build-up of particles in the ground state without a phase transition occurring. In this case there is no unique way to identify a temperature associated with the build-up of particles in the ground state. For the particular case of a constant magnetic field in three spatial dimensions it was shown by Rojas [32] that although there was no phase transition there could still be a build-up of particles in the ground state which could be interpreted as BEC. A similar situation occurs for bosons confined by a harmonic oscillator potential [34].

II. CRITERION FOR THE OCCURRENCE OF BOSE-EINSTEIN CONDENSATION

A. Relativistic quantum field theory

Let us consider a complex relativistic scalar field which may interact with background electromagnetic or gravitational fields but which is otherwise free. We restrict our attention to an ultrastatic spacetime of the form $\mathcal{M} = \mathbb{R} \times \Sigma$ with metric

$$ds^2 = dt^2 - g_{ij}(x)dx^i dx^j \tag{2.1}$$

and with the field obeying any boundary conditions in the spatial directions. The action functional for the complex field Φ will be chosen to be

$$S = \int dt \int_{\Sigma} d\sigma_x \ \{(D^\mu \Phi)^\dagger (D_\mu \Phi) - m^2 \Phi^\dagger \Phi - U_0(x)$$

$$-U_1(x)\Phi^\dagger\Phi\Big\}\,, \tag{2.2}$$

$d\sigma_x$ is the invariant volume element on the manifold Σ. In order to have a scalar field exclusively under external conditions, the functions $U_0(x)$ and $U_1(x)$ may depend on the background gravitational or electromagnetic fields, but are independent of the scalar field. They are also assumed to be independent of the time. The kinetic term is the usual gauge-covariant derivative

$$D_\mu\Phi = \partial_\mu\Phi - ieA_\mu\Phi. \tag{2.3}$$

In an ultrastatic spacetime, finite temperature and density are easily incorporated in the Euclidean time formalism [35–37]. It has been shown, that the partition function $\mathcal{Z}$ may be represented in the form (see for example [38])

$$\mathcal{Z} = \int [d\Phi][d\Phi^\dagger]\exp\left\{-\tilde{S}\right\}, \tag{2.4}$$

where

$$\begin{aligned}\tilde{S} = \int_0^\beta d\tau \int_\Sigma d\sigma_x\ \Big\{ &\left[\dot{\Phi}^\dagger + ie(A_0 - i\mu)\Phi^\dagger\right] \\ &\times\left[\dot{\Phi} - ie(A_0 - i\mu)\Phi\right] + |\mathbf{D}\Phi|^2 \\ &+(m^2 + U_1(x))\Phi^\dagger\Phi + U_0(x) + J\Phi + J^\dagger\Phi^\dagger\Big\}.\end{aligned} \tag{2.5}$$

In order to decide whether or not Bose-Einstein condensation will occur we need the charge in the excited states. It may be obtained directly from the effective action. Eliminating the dependence on the sources J and $J^\dagger$ by

$$\begin{aligned}\bar{\Phi} &= \frac{\delta W}{\delta J}\ |_{\mu,J^\dagger} \\ \bar{\Phi}^\dagger &= \frac{\delta W}{\delta J^\dagger}\ |_{\mu,J}\,,\end{aligned} \tag{2.6}$$

with the background fields $\bar{\Phi}$, $\bar{\Phi}^\dagger$, the effective action is defined by

$$\begin{aligned}\Gamma[\mu,\bar{\Phi},\bar{\Phi}^\dagger] &= W[\mu,J,J^\dagger] \\ &\quad - \int_0^\beta d\tau \int_\Sigma d\sigma_x\ [J\bar{\Phi} + J^\dagger\bar{\Phi}^\dagger]\end{aligned} \tag{2.7}$$

with

$$W[\mu, J, J^\dagger] = -\ln \mathcal{Z}[\mu, J, J^\dagger]. \tag{2.8}$$

It is known, that a minimization of the effective action is equivalent to a minimization of the Helmholtz free energy [25]. In terms of the effective action the charge is then defined by

$$Q = -\frac{1}{\beta}\frac{\partial \Gamma}{\partial \mu}\,|_{\bar{\Phi},\bar{\Phi}^\dagger}\,. \tag{2.9}$$

We may now proceed with the analysis of the effective action. It is straightforward to show that

$$\Gamma[\mu, \bar{\Phi}, \bar{\Phi}^\dagger] = \tilde{S}[\bar{\Phi}, \bar{\Phi}^\dagger] + \frac{1}{2}\ln\det(l^2 \tilde{S}_{,ij}), \tag{2.10}$$

where we used DeWitt's condensed notation [39]. l is an arbitrary unit of length introduced to keep the argument of the logarithm in (2.10) dimensionless. Specializing to the case of a background static magnetic field only, and choosing the gauge

$$A_0 = 0, \qquad \nabla \cdot \mathbf{A} = 0, \tag{2.11}$$

we have

$$\frac{1}{2}\ln\det(l^2 S_{,ij}) = \Gamma_+ + \Gamma_-, \tag{2.12}$$

where

$$\Gamma_\pm = \frac{1}{2}\ln\det\left\{ l^2 \left[-\left(\frac{\partial}{\partial \tau} \mp e\mu\right)^2 - \mathbf{D}^2 \right.\right. \\ \left.\left. + m^2 + U_1(x) \right]\right\}. \tag{2.13}$$

For the calculation of (2.13) we will use the zeta function scheme [40,41]. In this scheme one defines

$$\Gamma_\pm = -\frac{1}{2}\zeta_\pm'(0) + \frac{1}{2}\zeta_\pm(0)\ln l^2 \tag{2.14}$$

with the generalized zeta functions

$$\zeta_{\pm}(s) = \sum_{j=-\infty}^{\infty} \sum_{N} (\lambda_{jN}^{\pm})^{-s}. \tag{2.15}$$

Here, $\lambda_{jN}^{\pm}$ are the eigenvalues of the fluctuation operator (see (2.13)) given by

$$\lambda_{jN}^{\pm} = \left(\frac{2\pi}{\beta} j \pm ie\mu \right)^2 + \sigma_N, \tag{2.16}$$

with the eigenvalues σ_N of the spatial part of the Klein-Gordon operator,

$$(-\mathbf{D}^2 + U_1(x) + m^2) f_N(x) = \sigma_N f_N(x), \tag{2.17}$$

with a complete orthonormal set of functions $f_N(x)$. It is easily seen that $\Gamma_+ = \Gamma_-$, so that we skip the index $\pm$ in the notation and consider ζ_+.

Now all the definitions and preparation to explain our criterion for the appearance of Bose-Einstein condensation have been given. For the moment we do not want to consider any specific situation but want to assume a quite general structure of the eigenvalues λ_{jN} or, more specifically, of the energy $E_N^2 = \sigma_N$. As we will see, to decide if Bose-Einstein condensation appears, the only relevant information on E_N is about its continuous part. Let us assume that σ_N splits into the sum of a discrete part $\sigma_{\mathbf{p}}^d$ ($\mathbf{p}$ is just a set of labels for the discrete part of the spectrum), and a continuous part which we can deal with by imposing box normalization. The box will be taken to have sides $L_1, ..., L_q$, for some q and thus we write

$$E_N^2 = \sum_{i=1}^{q} \left(\frac{2\pi}{L_i} \right)^2 l_i^2 + \sigma_{\mathbf{p}}^d \, . \tag{2.18}$$

In the limit $L_i \to \infty$ the starting point for the analysis of $\zeta(s)$ is

$$\zeta(s) = \frac{V_q \beta}{(4\pi)^{\frac{q+1}{2}}} \frac{1}{\Gamma(s)} \sum_{l=-\infty}^{\infty} \sum_{\mathbf{p}} \int_0^{\infty} dt \; t^{s-1-\frac{q}{2}}$$
$$\times \exp\left\{ \left(\frac{2\pi i l}{\beta} - \mu \right)^2 t - \sigma_{\mathbf{p}}^d t \right\}. \tag{2.19}$$

Doing a resummation in l, this is equivalent to

$$\zeta(s) = \frac{V_q \beta}{(4\pi)^{\frac{q+1}{2}}} \frac{\Gamma\left(s - \frac{q+1}{2}\right)}{\Gamma(s)} \sum_{l=-\infty}^{\infty} \sum_{\mathbf{p}}$$

$$\times \int_0^\infty dt \; t^{s-1-\frac{q+1}{2}} e^{-\frac{\beta^2}{4t} l^2 - \sigma_{\mathbf{p}}^d t + \beta\mu l}$$

$$= \frac{V_q \beta}{(4\pi)^{\frac{q+1}{2}}} \frac{\Gamma\left(s - \frac{q+1}{2}\right)}{\Gamma(s)} \zeta_{d,\Sigma}\left(s - \frac{q+1}{2}\right)$$

$$+ \frac{V_q \beta}{(4\pi)^{\frac{q+1}{2}}} \frac{2}{\Gamma(s)} \sum_{l=1}^{\infty} \sum_{\mathbf{p}} \left(\frac{2}{\beta l}\sqrt{\sigma_{\mathbf{p}}^d}\right)^{\frac{q+1}{2}-s}$$

$$\times K_{\frac{q+1}{2}-s}(\beta l \sqrt{\sigma_{\mathbf{p}}^d}) \left(e^{\beta\mu l} + e^{-\beta\mu l}\right) \;, \tag{2.20}$$

where we introduced the zeta-function of the discrete part of the spatial section,

$$\zeta_{d,\Sigma}(s) = \sum_{\mathbf{p}} (\sigma_{\mathbf{p}}^d)^{-s}. \tag{2.21}$$

The condition for Bose-Einstein condensation to occur is that μ must reach a critical value μ_C set by the lowest eigenvalue in the spectrum,

$$\mu_C^2 = \sigma_{\mathbf{0}}^d = E_0^2. \tag{2.22}$$

If the charge Q in the excited states, Eq. (2.9), remains bounded as $\mu \to \mu_C$, then Bose-Einstein condensation occurs, because for the total charge large enough, it is not possible to accommodate it all in the excited states. If Q is not bounded as $\mu \to \mu_C$, then any amount of the total charge can reside in the excited states and Bose-Einstein condensation will not occur. We therefore need to look at the behaviour of $(\partial/\partial\mu)\zeta(0)$ and $(\partial/\partial\mu)\zeta'(0)$ as $\mu \to \mu_C$.

Using the definition (2.9) and representation (2.20) for $\zeta(s)$, one immediately finds

$$Q = V_q \frac{2}{(4\pi)^{\frac{q+1}{2}}} \sum_{l=1}^{\infty} \sum_{\mathbf{p}} \beta l \left(\frac{2}{\beta l}\sqrt{\sigma_{\mathbf{p}}^d}\right)^{\frac{q+1}{2}} K_{\frac{q+1}{2}}(\beta l \sqrt{\sigma_{\mathbf{p}}^d})$$
$$\times \left(e^{l\beta\mu} - e^{-l\beta\mu}\right). \tag{2.23}$$

The convergence of the sums is defined through the behaviour of the MacDonald functions for large arguments [45]

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left\{1 + \mathcal{O}(z^{-1})\right\}. \tag{2.24}$$

It is clear that in the limit $\mu \to \mu_C$ all but the contributions coming from $\mathbf{p} = \mathbf{0}$ are finite. Thus in the leading approximation, neglecting finite pieces, we write

$$Q(\mu \to \mu_C) = d_0 V_q \left(\frac{\sqrt{\sigma_{\mathbf{0}}^d}}{2\pi\beta}\right)^{q/2} \sum_{l=1}^{\infty} l^{-q/2} e^{-\beta l(\sqrt{\sigma_{\mathbf{0}}^d} - \mu)}. \tag{2.25}$$

We have allowed for the possible degeneracy of the ground state by introducing the degeneracy factor d_0. It is seen clearly that in the limit $\mu \to \mu_C = \sqrt{\sigma_{\mathbf{0}}^d}$ the charge remains finite for $q \geq 3$; thus for $q \geq 3$ Bose-Einstein condensation occurs, and for $q \leq 2$ Bose-Einstein condensation does not occur.

The detailed behaviour for $\mu \to \mu_C$ is most clearly extracted using the Mellin-Barnes integral representation of the exponential in (2.25),

$$e^{-v} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\alpha \; \Gamma(\alpha) v^{-\alpha}, \tag{2.26}$$

with $\Re v > 0$ and $c \in \mathbb{R}$, $c > 0$. Using this in Eq. (2.25), one finds

$$Q(\mu \to \mu_C) = \frac{d_0 V_q}{2\pi i} \left(\frac{\sqrt{\sigma_{\mathbf{0}}^d}}{2\pi\beta}\right)^{\frac{q}{2}} \int_{c-i\infty}^{c+i\infty} d\alpha \; \Gamma(\alpha)$$
$$\times (\sqrt{\sigma_{\mathbf{0}}^d} - \mu)^{-\alpha} \beta^{-\alpha} \zeta_R\left(\alpha + \frac{q}{2}\right), \tag{2.27}$$

where, in order to allow for interchanging the sum and the integral one has to impose $\alpha > 1 - q/2$. $\zeta_R(s)$ is the Riemann ζ-function, which is analytic in s except at $s = 1$ where it has a simple pole with residue 1. Closing the contour to the left of the rightmost pole then gives the following leading behaviour :

$q = 0$, pole of order one at $\alpha = 1$,

$$Q(\mu \to \mu_C) = \frac{d_0}{\beta(\mu_C - \mu)}, \tag{2.28}$$

$q = 1$, pole of order one at $\alpha = 1/2$,

$$Q(\mu \to \mu_C) = \frac{d_0 V_1}{\sqrt{2}\beta}\left(\frac{\mu_C}{\mu_C - \mu}\right)^{1/2}, \tag{2.29}$$

$q = 2$, pole of order two at $\alpha = 0$,

$$Q(\mu \to \mu_C) = -\frac{d_0 V_2}{2\pi\beta}\mu_C \ln \beta(\mu_C - \mu). \tag{2.30}$$

As mentioned, for $q \geq 3$ no divergent contribution results. The restriction $q \geq 3$ includes a large number of previously known results, often established by long and detailed calculations, as special cases. These and some new examples are summarized in Sec. 3.

B. Nonrelativistic quantum field theory

We will consider a nonrelativistic field theory described by the complex Schrödinger field Φ whose action functional is

$$\begin{aligned} S = \int dt \int_\Sigma d\sigma_x \Big\{ & \frac{i}{2}\left(\Phi^\dagger \dot{\Phi} - \dot{\Phi}^\dagger \Phi\right) - \frac{1}{2m}|\mathbf{D}\Phi|^2 \\ & -U_1(\mathbf{x})\Phi^\dagger\Phi \Big\} . \end{aligned} \tag{2.31}$$

$d\sigma_x$ is the invariant volume element on the D-dimensional Riemannian manifold Σ. We will consider Σ to be compact. (In the case $\Sigma = \mathbb{R}^D$ we will impose box normalization with the infinite box limit taken.) $U_1(\mathbf{x})$ is an arbitrary time independent potential. As before, $\mathbf{D} = \nabla - ie\mathbf{A}$ is the gauge covariant derivative, with $\mathbf{A}$ a vector potential describing a background magnetic field and $|\mathbf{D}\Phi|^2$ is short for $g^{ij}D_i\Phi^\dagger D_j\Phi$ where g_{ij} is the Riemannian metric on Σ.

The theory described by (2.31) has the local gauge invariance

$$\begin{aligned} \Phi &\to e^{ie\theta}\Phi , \\ \mathbf{A} &\to \mathbf{A} + \nabla\theta , \end{aligned} \tag{2.32}$$

which gives rise to a conserved current, and a conserved charge which is given by

$$Q = \int_\Sigma d\sigma_x |\Phi|^2 \,. \tag{2.33}$$

We will deal with the charge rather than the particle number to mirror the relativistic case considered above as closely as possible. The conserved charge is dealt with by introducing a Lagrange multiplier μ, which is the chemical potential. After a rotation to imaginary time τ, the partition function is again expressed in the form (2.4), with

$$\begin{aligned} \tilde{S}[\bar{\Phi}, \bar{\Phi}^\dagger] = \int_0^\beta d\tau \int_\Sigma d\sigma_x \Big\{ & \frac{1}{2}\left(\Phi^\dagger \frac{\partial}{\partial\tau}\Phi - \frac{\partial}{\partial\tau}\Phi^\dagger\Phi\right) \\ & + \frac{1}{2m}|\mathbf{D}\Phi|^2 + U_1(\mathbf{x})|\Phi|^2 \\ & - e\mu|\Phi|^2 + J\Phi + J^\dagger\Phi^\dagger \Big\} \,. \end{aligned} \tag{2.34}$$

The field Φ has been coupled to a complex source J as in (2.5). The effective action can now be defined as in Sec. 2.1. In place of (2.10) we find

$$\begin{aligned} \Gamma[\mu, \bar{\Phi}, \bar{\Phi}^\dagger] = \tilde{S}[\bar{\Phi}, \bar{\Phi}^\dagger] + \ln\det l \Big[& \frac{\partial}{\partial\tau} - e\mu \\ & - \frac{1}{2m}\mathbf{D}^2 + U_1(\mathbf{x}) \Big] \,, \end{aligned} \tag{2.35}$$

where $\bar{\Phi}$ is the background Schrödinger field. The second term in (2.35) has arisen from performing the functional integral in (2.4) over the Schrödinger field.

In order to regularize the determinant which appears in (2.35) we will again use the ζ-function method. This time we will let $f_N(\mathbf{x})$ denote the eigenfunctions of $-\frac{1}{2m}\mathbf{D}^2 + U_1(\mathbf{x})$:

$$\left[-\frac{1}{2m}\mathbf{D}^2 + U_1(\mathbf{x}) \right] f_N(\mathbf{x}) = \sigma_N f_N(\mathbf{x}) \,. \tag{2.36}$$

(These $f_N(\mathbf{x})$ differ only by a trivial rescaling from the $f_N(\mathbf{x})$ used in Sec. 2.1.) The $f_N(\mathbf{x})$ are seen to be stationary state solutions to the Schrödinger equation for whatever boundary conditions are imposed. The eigenvalues σ_N are the energy levels for the first quantized system. The set $\{f_N(\mathbf{x})\}$ is assumed to be complete and orthonormal.

Because the functional integral in (2.4) extends over all fields periodic in the imaginary time coordinate τ with period $\beta = T^{-1}$, the eigenvalues of the operator appearing in (2.35) are

$$\lambda_{jN} = 2\pi i j T + \sigma_N - e\mu \ , \tag{2.37}$$

where $j = 0, \pm 1, \pm 2, \ldots$. The generalized ζ-function is given by (see 2.15)

$$\zeta(s) = \sum_{j=-\infty}^{\infty} \sum_N (\lambda_{jN})^{-s} \ , \tag{2.38}$$

and we define

$$\ln \det l \left[\frac{\partial}{\partial \tau} - e\mu - \frac{1}{2m}\mathbf{D}^2 + U_1(\mathbf{x}) \right]$$

$$= -\zeta'(0) + \zeta(0) \ln l \ . \tag{2.39}$$

Using the summation formula in [42], it is easy to show that

$$\zeta(s) = \sum_N (\sigma_N - e\mu)^{-s} + \zeta_T(s) \ , \tag{2.40}$$

where we have defined

$$\zeta_T(s) = \frac{T^{-s}}{\Gamma(s)} \sum_N \sum_{n=1}^{\infty} \frac{e^{-n\beta(\sigma_N - e\mu)}}{n^{1-s}} \ . \tag{2.41}$$

The first term in (2.40) has no explicit temperature dependence, and corresponds to the zero-point energy contribution to the effective action. We will ignore it in what follows. It disappears if normal ordering of the Hamiltonian is adopted.

The term in $\zeta_T(s)$, which we may call the thermal ζ-function, is easily shown to have the following values :

$$\zeta_T(0) = 0 \ , \tag{2.42}$$

$$\zeta_T'(0) = -\sum_N \ln\left[1 - e^{-\beta(\sigma_N - e\mu)}\right] \ . \tag{2.43}$$

These last two results show that the second term in (2.35) has a simple relation to the thermodynamic potential Ω defined by

$$\Omega = T \sum_N \ln\left[1 - e^{-\beta(\sigma_N - e\mu)}\right] \ . \tag{2.44}$$

(We could have adopted Ω as our starting point; however, our criterion is most simply expressed using the generalized ζ-function, and in addition we wished to parallel the relativistic calculation.)

We will again assume that σ_N splits up into the sum of a discrete part $\sigma^d_{\mathbf{p}}$, and a continuous part which is dealt with by box normalization as in (2.18) with E_N^2 replaced by σ_N. In the large box limit, the thermal ζ-function becomes

$$\zeta_T(s) = \frac{V_q}{(4\pi)^{q/2}} \frac{T^{q/2-s}}{\Gamma(s)} \sum_{\mathbf{p}} \sum_{n=1}^{\infty} \frac{e^{-n\beta(\sigma^d_{\mathbf{p}} - e\mu)}}{n^{1+q/2-s}} \ . \tag{2.45}$$

As in the relativistic case, the lowest mode $\sigma^d_{\mathbf{0}}$ plays the crucial role in determining whether or not Bose-Einstein condensation can occur. We therefore split the sum over $\mathbf{p}$ in (2.45) by separating off the lowest mode for special treatment. The critical value of the chemical potential is given by

$$e\mu_C = \sigma_0 = \sigma^d_{\mathbf{0}} \ , \tag{2.46}$$

and we wish to study the behaviour of the charge as $\mu \to \mu_C$. If we write

$$\zeta_T(s) = \zeta_T^{(0)}(s) + \zeta_T^{(\neq 0)}(s) \ , \tag{2.47}$$

where

$$\zeta_T^{(0)}(s) = \frac{d_0 V_q}{(4\pi)^{q/2}} \frac{T^{q/2-s}}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{e^{-n\beta e(\mu_C - \mu)}}{n^{1+q/2-s}} \tag{2.48}$$

is the lowest mode contribution (with degeneracy d_0), and $\zeta_T^{(\neq 0)}(s)$ is given by (2.45) with the sum over $\mathbf{p}$ restricted to non-zero values, it is easy to see that because the argument of the exponential in (2.45) will always be negative, even for $\mu = \mu_C$, $\zeta_T^{(\neq 0)\prime}(0)$ and $\frac{\partial}{\partial \mu}\zeta_T^{(\neq 0)\prime}(0)$ remain finite. Bose-Einstein condensation is therefore determined from a knowledge of $\zeta_T^{(0)}(s)$.

From (2.48) we have $\zeta^{(0)}(0) = 0$ and

$$\zeta_T^{(0)\prime}(0) = \frac{d_0 V_q}{(4\pi)^{q/2}} T^{q/2-s} \sum_{n=1}^{\infty} \frac{e^{-n\beta e(\mu_C - \mu)}}{n^{1+q/2}} \ . \tag{2.49}$$

The charge is given by

$$Q = -T\frac{\partial}{\partial\mu}\Gamma = T\frac{\partial}{\partial\mu}\zeta_T^{(0)\prime}(0) + \cdots \ , \tag{2.50}$$

where terms which remain finite as $\mu \to \mu_C$ have been dropped. Using (2.49) we have

$$Q(\mu \to \mu_C) \simeq ed_0 V_q \left(\frac{T}{4\pi}\right)^{q/2} \sum_{n=1}^{\infty} \frac{e^{-n\beta e(\mu_C-\mu)}}{n^{q/2}} \ . \tag{2.51}$$

At $\mu = \mu_C$ the sum appearing in this expression for the total charge diverges for $q \leq 2$. This gives us the necessary and sufficient condition for Bose-Einstein condensation to occur. For $q \geq 3$, Bose-Einstein condensation does not occur.

We can obtain more detailed information on how Q diverges as $\mu \to \mu_C$ as before by making use of (2.26). It is easily seen that (2.51) becomes

$$\begin{aligned} Q(\mu \to \mu_C) = \frac{ed_0V_q}{2\pi i}\left(\frac{T}{4\pi}\right)^{q/2} \int_{c-i\infty}^{c+i\infty} d\alpha\Gamma(\alpha) \\ \times \left(\frac{T}{\mu_C-\mu}\right)^{\alpha} \zeta_R(\alpha+q/2) \ . \end{aligned} \tag{2.52}$$

This result may be used to show that

$$Q(\mu \to \mu_C) \simeq \frac{eTd_0}{\mu_C-\mu} \quad (q=0); \tag{2.53}$$

$$Q(\mu \to \mu_C) \simeq \frac{1}{2}ed_0V_1T(\mu_C-\mu)^{-1/2} \quad (q=1); \tag{2.54}$$

$$Q(\mu \to \mu_C) \simeq \frac{eTd_0V_2}{4\pi}\ln\left(\frac{T}{\mu_C-\mu}\right) \quad (q=2). \tag{2.55}$$

Only the leading part of Q which diverges as $\mu \to \mu_C$ has been shown in these expressions. The way in which Q diverges as $\mu \to \mu_C$ is seen to be the same for relativistic and nonrelativistic systems.

III. APPLICATION OF THE CRITERION TO EXAMPLES

In this section we will consider several examples of an ideal Bose gas under possible external conditions and we will see that the conclusion whether or not Bose-Einstein condensation might occur can be drawn very easily using our criterion of section 2.

Let us start with the free ideal Bose gas in a $(D+1)$-dimensional Minkowski space-time. Then the eigenvalues of the Klein-Gordon operator (2.17) are

$$E_{\vec{k}}^2 = \vec{k}^2 + m^2, \qquad \vec{k} \in \mathbb{R}^D. \tag{3.1}$$

In the non-relativistic case we have

$$E_{\vec{k}} = \frac{1}{2m}\vec{k}^2, \qquad \vec{k} \in \mathbb{R}^D . \tag{3.2}$$

In either case we see that $q = D$, and conclude that BEC can only occur for $D \geq 3$. This agrees with conclusions of Refs. [6,7,5] for the relativistic gas, and Refs. [23,43] for the non-relativistic gas.

There are many possibilities of restricting the Minkowski space by imposing boundary conditions in one or more directions. For example imposing Dirichlet boundary conditions in one direction, $f_N(x_i = 0) = f_N(x_i = L)$, the energy eigenvalues eq. (2.18) are

$$E_{n,\vec{k}}^2 = \vec{k}^2 + \left(\frac{\pi n}{L}\right)^2 + m^2,\ \vec{k} \in \mathbb{R}^{D-1},\ n \in \mathbb{N}, \tag{3.3}$$

for the relativistic field, and

$$E_{n,\vec{k}} = \frac{1}{2m}\left\{\vec{k}^2 + \left(\frac{\pi n}{L}\right)^2\right\},\ \vec{k} \in \mathbb{R}^{D-1},\ n \in \mathbb{N}, \tag{3.4}$$

for the non-relativistic field. In this case we find $q = D - 1$ and therefore BEC is only expected for $D \geq 4$ (corresponding to $q \geq 3$). If we impose Dirichlet boundary conditions in p of the spatial dimensions then $q = D - p$ and BEC would only be expected for $D \geq 3 + p$. The choice of Dirichlet boundary conditions is not important here, and periodic or Neumann boundary conditions would lead to the same result. The restriction $D \geq 3 + p$ holds for any choice of boundary conditions which results in a discrete spectrum in p spatial dimensions.

Another important case where the energy spectrum contains a discrete part is found when there is a constant external magnetic field present. Starting with a single component constant magnetic field B one encounters the well known Landau levels giving

$$E_{n,\vec{k}_\perp}^2 = (2n+1)eB + \vec{k}_\perp^2 + m^2,\ n \in \mathbb{N}_0,\ \vec{k}_\perp \in \mathbb{R}^{D-2}, \tag{3.5}$$

for the relativistic field, and

$$E_{n,\vec{k}_\perp} = \frac{1}{2m}\left\{(2n+1)eB + \vec{k}_\perp^2\right\} \tag{3.6}$$

in the nonrelativistic case. In both cases the eigenvalues are degenerate with degeneracy $(SeB)/(2\pi)$ with the (formal) volume S of $\mathbb{R}^2$. The single component magnetic field results in $q = D-2$, and condensation can occur only for $D \geq 5$. This was shown for the relativistic field in [27,28] and for the non-relativistic field in [24]. In particular BEC is absent for $D = 3$ [22].

In the general case, a magnetic field in D spatial dimensions is characterized by δ independent components, where $D = 2\delta$ or $2\delta + 1$ depending upon whether D is even or odd [26,38]. It is straightforward to show that

$$E^2_{n,\vec{k}_\perp} = \sum_{j=1}^{l}(2n_j+1)eB_j + \vec{k}_\perp^2 + m^2, \tag{3.7}$$

for the relativistic field, and

$$E_{n,\vec{k}_\perp} = \frac{1}{2m}\left\{\sum_{j=1}^{l}(2n_j+1)eB_j + \vec{k}_\perp^2\right\}, \tag{3.8}$$

for the non-relativistic field. In both cases $n_j \in \mathbb{Z}_0$ and $\vec{k}_\perp \in \mathbb{R}^{D-2l}$. The degeneracy is $\prod_{j=1}^{p} eB_j L_{2j-1}L_{2j}/(2\pi)$ where B_j $j = 1, \ldots, p$ represent the independent field strengths and $p \leq \delta$. (L_j is the length of the box in the j^{th} direction with $L_j \to \infty$ assumed.) In either case, $q = D - 2p$, so that every component of the magnetic field reduces the effective dimension by 2. The criterion for BEC is $D \geq 3 + 2p$ in agreement with the explicit calculations in Refs. [26,38].

Another example is to consider a field on the space $\mathbb{S}^p \times \mathbb{R}^{D-p}$ where $\mathbb{S}^p$ is the p-dimensional sphere. In this case we have

$$E^2_{n,\vec{k}} = n(n+p-1)a^{-2} + \vec{k}^2 + \xi a^{-2} + m^2 \tag{3.9}$$

for the relativistic field, and

$$E_{n,\vec{k}} = \frac{1}{2m}\left[n(n+p-1)a^{-2} + \xi a^{-2} + \vec{k}^2\right], \tag{3.10}$$

for the non-relativistic field. a represents the radius of the sphere, and ξ is a dimensionless coupling constant which is related to a possible coupling of the scalar field to the scalar curvature. We have $q = D - p$ here, and BEC requires $D \geq 3 + p$. In the case of greatest physical interest we have $p = D = 3$ which corresponds to the Einstein static universe. BEC does not occur in this case as shown recently in Ref. [44].

In the examples we have discussed so far the energy spectrum has been explicitly known so that a direct evaluation of the thermodynamics has been possible. Our next example is to consider a case where the spectrum is not explicitly known; namely a single component constant magnetic field in a cylindrical box of radius R, where the field satisfies Dirichlet boundary conditions [38]. The eigenvalues for this problem are given in implicit form only and the equation reads

$$M\left(\frac{n}{2} - \frac{1}{4}\gamma, n+1; \frac{1}{2}eBR^2\right) = 0 \tag{3.11}$$

with $n \in \mathbb{N}$, $\gamma = (2/eB)[E_N - eB(n+1) - \vec{k}_\perp^2]$, $\vec{k}_\perp^2 \in \mathbb{R}^{D-2}$, and $M(a, b; c)$ the confluent hypergeometric function [45]. The zeroes of (3.11) are a discrete set of values for γ; thus $q = D - 2$ and condensation occurs for $D \geq 5$, a criterion derived without any problem. The exact analysis of the problem would be impossible due to the lack of an explicit form for the zeros of the confluent hypergeometric function.

A similar situation to the last example occurs if we consider a gas confined by an arbitrarily shaped cavity of dimension p in a space of dimension D. It is impossible to write down any explicit form for the energy eigenvalues. If the confinement is achieved by imposing Dirichlet boundary conditions say, then the portion of the energy spectrum coming from the cavity would be discrete and $q = D - p$. A more complete analysis of this problem will be given elsewhere [46].

As our final example we consider a non-relativistic gas in a harmonic oscillator potential. If we assume that

$$U(\mathbf{x}) = \frac{1}{2}m\sum_{j=1}^{p}\omega_j^2 x_j^2 \ , \tag{3.12}$$

with $p \leq D$, then

$$E_{n,\vec{k}} = \frac{1}{2m}\left\{\vec{k}^2 + \sum_{j=1}^{p}(n_j + \frac{1}{2})\omega_j\right\} \ , \quad \vec{k} \in \mathbb{R}^{D-p} \ , \tag{3.13}$$

are the energy levels. We have $q = D - p$ here so that $D \geq 3 + p$ is required for BEC. In the case of greatest physical interest $D = 3$ it can be seen that BEC does not occur. At this stage it is worth repeating the comment in the last paragraph of Sec. 1. We are interpreting BEC in the sense of symmetry breaking with an associated phase transition, such as that which occurs for the free Bose gas with $D = 3$. This does not occur unless $D \geq 3 + p$. We are not saying that there cannot be a build-up of particles in the ground state unassociated with a phase transition as the temperature is lowered.

This concludes the list of examples for the relativistic and non-relativistic Bose gas. As we have seen, known examples and in addition new examples can be dealt with very easily.

IV. APPEARANCE OF THE MEISSNER-OCHSENFELD EFFECT FOR A GENERAL HOMOGENEOUS MAGNETIC FIELD

A. Relativistic scalar field in an external magnetic field

Let us consider the situation where a general homogeneous magnetic field is present. The energy eigenvalues then read

$$E_N^2 = \sum_{j=1}^{p}(2n_j + 1)\nu_j + \sum_{j=2p+1}^{D}\left(\frac{2\pi l_j}{L_j}\right)^2 + m^2 \tag{4.1}$$

with the limit $L_j = L \to \infty$ taken and $\nu_j = eB_j$. p is the number of non-zero components of the magnetic field as discussed in the last section. The degeneracy for each Landau level is $(L^2/(2\pi))^p \prod_{j=1}^{p}\nu_j$. (This degeneracy corresponds to d_0 introduced in Sec. 2.) Thus for the zeta function we have

$$\zeta(s) = \frac{L^D}{2^{D-p}\pi^{D/2}\Gamma(s)}\prod_{j=1}^{p}\nu_j \sum_{l=-\infty}^{\infty}\sum_{n_1,\ldots,n_p=0}^{\infty} \times \tag{4.2}$$

$$\int_0^\infty dt\ t^{s-1+p-D/2} e^{\left(\frac{2\pi i l}{\beta}-\mu\right)^2 t - \sum_{j=1}^{p}(2n_j+1)\nu_j t - m^2 t}.$$

For the magnetization

$$M_i = -\frac{1}{L^D\beta}\frac{\partial}{\partial B_i}\Gamma \tag{4.3}$$

we need $(\partial/\partial B_i)\zeta(s)$. (The magnetization and the choice of units is described in the Appendix.) Using the same kind of steps as for the derivation of (2.20), we find

$$\begin{aligned}
\frac{\partial}{\partial B_i}\zeta(s) = & \frac{L^D\beta}{2^{D+1-p}\pi^{\frac{D+1}{2}}\Gamma(s)}\left(\prod_{j=1}^{p}\nu_j\right) \\
& \times\left\{-e\Gamma\left(s-\frac{q-1}{2}\right)\tilde{\zeta}_{d,\Sigma}\left(\frac{q-1}{2}-s\right)\right. \\
& \left.+\frac{1}{B_i}\Gamma\left(s-\frac{q+1}{2}\right)\zeta_{d,\Sigma}\left(\frac{q+1}{2}-s\right)\right\} \\
& +\frac{L^D\beta}{2^{\frac{D+1}{2}+s}\pi^{\frac{D+1}{2}}\Gamma(s)}\prod_{j=1}^{p}\nu_j\sum_{l=1}^{\infty}\sum_{n_1,\ldots,n_p=0}^{\infty}e^{\beta\mu l} \\
& \times\left(\frac{\sqrt{\sigma_{\mathbf{p}}^d}}{\beta l}\right)^{\frac{q-1}{2}-s}\left\{-e(2n_i+1)\right. \\
& \times K_{\frac{q-1}{2}-s}(\beta l\sqrt{\sigma_{\mathbf{p}}^d}) \\
& \left.+\frac{2}{B_i}\left(\frac{\sqrt{\sigma_p^d}}{\beta l}\right)K_{\frac{q+1}{2}-s}(\beta l\sqrt{\sigma_{\mathbf{p}}^d})\right\} \\
& +(\mu\to-\mu),
\end{aligned} \tag{4.4}$$

where we defined

$$\tilde{\zeta}_{d,\Sigma}(s) = \sum_{n_1,\ldots,n_p=0}^{\infty}(2n_i+1)(\sigma_{\mathbf{p}}^d)^{-s}, \tag{4.5}$$

and abbreviated to $(\mu\to-\mu)$ the terms obtained from replacing μ in the second set of terms in (4.4) with $-\mu$. In order to decide whether or not the Meissner-Ochsenfeld effect takes place we can consider the limit $B_i\to 0$ and see if a nonvanishing magnetization results. This is equivalent to the generalization of the Meissner-Ochsenfeld effect from $D=3$ dimensions [30] to arbitrary dimension D. When $B_i\to 0$ this removes one of the discrete degenerate Landau levels and replaces it with two additional continuous labels. (See eq. (4.1).) The effect of this is to increase q by 2 to $q_{eff} = q+2$. Our general analysis showed that the

gas could only condense for $q \geq 3$. Thus if $q_{eff} \geq 3$, or $q = 1, 2$ there may be interesting behaviour when a magnetic field is present even though strict BEC cannot occur. The case $q = 1$ in the limit $B_i \to 0$ can be compared with the gas for $B_i = 0$ with $q = 3$; the case $q = 2$ in the limit $B_i \to 0$ can be compared with the gas for $B_i = 0$ with $q = 4$. In either case we wish to study the behaviour of the magnetization as μ becomes close to the critical value μ_c. As argued in our criterion for BEC, the leading term comes from the smallest energy eigenvalue and it reads

$$M_i(B_i \to 0) = \frac{e\beta^{-q/2}}{2^{\frac{D+2}{2}}\pi^{D/2}} \left(\prod_{j=1}^{p} \nu_j \right) \mu^{\frac{q-2}{2}} \times \sum_{l=1}^{\infty} l^{-q/2} e^{-\beta l(\mu_C - \mu)}, \tag{4.6}$$

which is very similar to the behaviour of the charge, Eq. (2.25). We see that for $q \geq 3$ the sum is convergent, even for $\mu = \mu_C$, and thus $M_i(B_i \to 0) = 0$ for that case. However, for $q = 1, 2$, we find using once more (2.26),

$q = 1$

$$M_i(B_i \to 0) = \frac{eT}{2^{\frac{D+2}{2}}\pi^{\frac{D-1}{2}}} \left(\prod_{j=1}^{p} \nu_j \right) \frac{1}{\sqrt{\mu_C}\sqrt{\mu_C - \mu}}, \tag{4.7}$$

$q = 2$

$$M_i(B_i \to 0) = -\frac{eT}{2^{\frac{D+2}{2}}\pi^{\frac{D}{2}}} \left(\prod_{j=1}^{p} \nu_j \right) \ln(\mu_C - \mu). \tag{4.8}$$

For $q = 0$ in the limit $B_i \to 0$ the effective dimension gets increased to $q = 2$ and thus no condensation can occur. Thus $\mu < \mu_C$, the sums are convergent and though $M_i(B_i \to 0) = 0$ in that case in the given approximation.

The magnetization laws, Eqs. (4.7) and (4.8), strongly depend on how $\mu \to \mu_C$ for $B_i \to 0$. It is clear, that the charge in the continuum above the lowest Landau level will condense into the ground state in the limit $B_i \to 0$. Let us use the notation $Q_{gr}(\mu \to \mu_C)$ for that charge. This is however exactly the charge given in Eqs. (2.29) and (2.30), and

the limit $M_i(B_i \to 0)/Q_{gr}(\mu \to \mu_C)$ is seen to be finite. If we introduce the charge density $\rho_{gr} = Q_{gr}/L^D$, it reads for $q = 1, 2$,

$$\frac{M_i(B_i \to 0)}{\rho_{gr}(\mu \to \mu_C)} = -\frac{e}{2\mu_C}. \tag{4.9}$$

Using the universal knowledge of the charge Q_{gr} in the ground state as a function of the critical temperature [12,13],

$$Q_{gr} = Q\left(1 - \left(\frac{T}{T_c}\right)^{D-1}\right) \tag{4.10}$$

with the total charge Q, and total charge density $\rho = Q/L^D$, we arrive at

$$M_i(B_i \to 0) = -\frac{e}{2\mu_C}\rho\left(1 - \left(\frac{T}{T_c}\right)^{D-1}\right), \tag{4.11}$$

with $\mu_C = \sqrt{m^2 + \sum_{j=1,j\neq i}^{p} \nu_j}$ as the magnetization law for $q = 1, 2$. For $q = 1$ in three spatial dimensions ($D = 3$) the result reduces to the one of ref. [30], and we agree with their conclusions on the Meissner-Ochsenfeld effect.

B. Non-relativistic scalar field

The energy levels for a p-component constant magnetic field are

$$E_{n,l} = \sum_{j=1}^{p}(2n_j + 1)\frac{eB_j}{2m} + \frac{1}{2m}\sum_{j=2p+1}^{D}\left(\frac{2\pi l_j}{L_j}\right)^2 , \tag{4.12}$$

where $n_j = 0, 1, \ldots$, and $L_j = L \to \infty$ as before. The degeneracy is $L^{2p}\prod_{j=1}^{p} eB_j/(2\pi)$. The effective action is determined as described in Sec. 2.2, and the magnetization found as in (4.3). We can again argue that $q = 1, 2$ are the interesting values as $B_i \to 0$ since in this limit we obtain $q_{eff} = q + 2 = 3, 4$ for which BEC is possible. The lowest state contribution is of greatest interest in this limit and we find

$$\begin{aligned} M_i(B_i \to 0) \simeq{}& -\frac{e}{2m}\left(\frac{m}{2\pi\beta}\right)^{q/2}\left(\frac{eB_j}{2\pi}\right) \\ &\times \sum_{n=1}^{\infty}\frac{e^{-n\beta e(\mu_c - \mu)}}{n^{q/2}} , \end{aligned} \tag{4.13}$$

as the leading contribution. (Note that $\mu_c = \frac{1}{2m}\sum_{j=1}^{p} eB_j$ here.) Apart from a numerical prefactor, this is the same as in the relativistic case in (4.6). For $q \geq 3$ the sum converges and we find $M_i \to 0$ as $B_i \to 0$ even if $\mu = \mu_c$. The sum occurring in (4.13) is the same as that occurring in the ground state charge (2.51).. If we write $\rho_{gr} = Q(\mu \simeq \mu_c)/L^D$ as the ground state charge density we find

$$M_i(B_i \to 0) \simeq -\frac{\rho_{gr}}{2m} \, . \tag{4.14}$$

This is essentially the same as the result found by Schafroth [22] for $D = 3$.

V. CONCLUSIONS

In this article we considered the general setting of a quantum field, relativistic or nonrelativistic, without self-interactions under general external conditions. We tried to answer the question whether or not the system might undergo a Bose-Einstein condensation, where by Bose-Einstein condensation we mean a phase transition associated with a build-up of particles in the ground state. Our main result is that the crucial feature governing Bose-Einstein condensation is the dimension q associated with the continuous part of the eigenvalue spectrum of the Hamiltonian for nonrelativistic systems or the spatial part of the Klein-Gordon operator for relativistic systems. In either case Bose-Einstein condensation can only occur if $q \geq 3$. Having this criterion at hand, many of previous results were obtained easily in section 3. In addition some new applications were given. The criterion was shown by studying the lowest eigenvalue of a differential operator. This relates to the idea of the effective infrared dimensions and finite size effects studied earlier [47].

Furthermore, we applied our ideas to the appearance of the Meissner-Ochsenfeld effect for a general homogeneous magnetic field. With the help of our criterion a discussion of this effect is greatly simplified and special cases are easily recovered [22,30,31].

The extension of our method to study Bose-Einstein condensation in theories with self-interaction [9,10] is of obvious interest. Recently we have shown how this problem can be

tackled very efficiently using ζ-function methods [11]. It is very likely that the method described in the present paper can be extended to interacting field theory. In particular it is possible to define an effective field theory describing the lowest modes as in ref. [47].

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APPENDIX: A NOTE ABOUT UNITS

As in classical electromagnetism, the magnetization $\mathbf{M}$ causes an effective current density which changes the effective field strength. This was noted by Schafroth [22] in the non-relativistic case, who argued that because the radius of the classical orbit for the charged particles was much greater than the average interparticle separation (whose scale is set by the particle density), the effective field strength should be identified with the average microscopic field. In the opposite case, where the interparticle separation is much greater than the classical charged particle orbit, then the correct procedure is to treat the particles as point dipoles.

An important point is that Schafroth used cgs units for his electromagnetic field conventions. In this case (see Jackson [48]) the magnetization gives rise to an effective current density $\mathbf{J}_M = c\nabla \times \mathbf{M}$ which must be added to the current in the Maxwell equation $\nabla \times \mathbf{B} = \dfrac{4\pi}{c}\mathbf{J}$. The resulting Maxwell equation may be written as $\nabla \times \mathbf{H} = \dfrac{4\pi}{c}\mathbf{J}$ where

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \, . \tag{A1}$$

An important point is that the factor of 4π which enters here has nothing to do with the fact that we are in three spatial dimensions. It is simply a consequence of the fact that cgs units have been chosen. In SI units we make the changes [48]

$$\mathbf{B} \rightarrow \sqrt{\frac{4\pi}{\mu_0}}\,\mathbf{B}$$

$$\mathbf{M} \to \sqrt{\frac{\mu_0}{4\pi}}\,\mathbf{M}$$
$$\mathbf{H} \to \sqrt{4\pi\mu_0}\,\mathbf{H}$$

and (A1) is changed to

$$\mathbf{H} = \frac{\mathbf{1}}{\boldsymbol{\mu_0}}\mathbf{B} - \mathbf{M}\,. \tag{A2}$$

In this case there is no factor of 4π in front of $\mathbf{M}$.

Our reason for making such a fuss over the units is that Ref. [27,28] has altered (A1) to read

$$\mathbf{H} = \mathbf{B} - \frac{\mathbf{D}\pi^{\mathbf{D/2}}}{\boldsymbol{\Gamma}(\mathbf{1}+\mathbf{D/2})}\mathbf{M} \tag{A3}$$

in D spatial dimensions. This expression first appears in the work of May [24] who generalized the original calculation of Schafroth [22] from 3 to D dimensions. The D-dependent factor in (A3) was noted as the area of the $(D-1)$-dimensional sphere by May. However, it should be clear that the factor appearing in front of the magnetization $\mathbf{M}$ is due to the choice of units, nothing more. (For the purist, we note that all of the above can be rewritten in terms of differential forms with the exterior derivative $\mathbf{d}$ in place of $\nabla\times$.)

There is another simple way of deducing the effective magnetic field which bears out our interpretation. The energy stored in the magnetic field is

$$W = \frac{1}{8\pi}\int_{\Sigma} d\sigma_x\,\mathbf{B}\cdot\mathbf{B} \tag{A4}$$

in cgs units [48]. If $\mathbf{B}$ is constant, then we have simply

$$W = \frac{1}{8\pi}VB^2\,, \tag{A5}$$

with $B = |\mathbf{B}|$. This term should really be added to the free energy F computed from the partition function or thermodynamic potential to form

$$F_T = F + \frac{1}{8\pi}VB^2\,. \tag{A6}$$

The magnetization is identified by the usual thermodynamic expression [49] to be

$$M = -\frac{1}{V}\frac{\partial F}{\partial B}\,, \tag{A7}$$

and therefore we have

$$\frac{\partial F_T}{\partial B} = \frac{V}{8\pi}(B - 4\pi M)\,. \tag{A8}$$

In our calculation, $M = |\mathbf{M}|$ has arisen from one-loop quantum effects. In the absence of quantum effects we would have simply the term in (A8) which involves B. This suggests that we define the effective field strength as in (A1). If SI units are adopted the factors of $\frac{1}{8\pi}$ in (A4–A6) are replaced with $\frac{1}{2\mu_0}$, and a repeat of this simple argument identifies (A2) as the effective field strength.

In relativistic quantum field theory it is conventional to use neither cgs nor SI units, but instead Heaviside-Lorentz rationalized units. In these units, where the lagrangian density for electromagnetism is $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, (A6) becomes

$$F_T = F + \frac{1}{2}VB^2\,, \tag{A9}$$

and we identify

$$H = B - M \tag{A10}$$

as the effective field strength. (The difference between Heaviside-Lorentz rationalized units and cgs units is that $B_{\rm cgs} = \sqrt{4\pi}\,B_{\rm HL}$, with a similar scaling relation for the vector gauge fields A_μ. See Schweber [50] for example.) We adopt Heaviside-Lorentz rationalized units in our work, as usual in quantum field theory.

We will now show how the effective action formalism may be used to calculate the magnetization for a general applied magnetic field in a space of arbitrary dimension. The full effective action may be written as

$$\Gamma = S_{\rm em} + \tilde{S} + \tilde{\Gamma} \tag{A11}$$

where

$$S_{\rm em} = \beta \int_\Sigma d\sigma_x \left\{ \frac{1}{4} F_{ij} F^{ij} - J^i_{\rm ext} A_i \right\} \tag{A12}$$

is the electromagnetic field action in Heaviside-Lorentz rationalized units. F_{ij} is the magnetic field tensor in a Riemannian space Σ with the time taken to be imaginary with periodicity β. We assume that the magnetic field is static, but otherwise arbitrary at this stage. $J^i_{\rm ext}$ is the externally applied current responsible for setting up the magnetic field. $\tilde{S}$ represents the contribution from the background scalar field, or Schrödinger field. $\tilde{\Gamma}$ represents the contribution from quantum effects of the matter fields.

We can settle the question of the correct expression for the magnetization in D spatial dimensions by working out the Maxwell equations for the magnetic field with quantum effects included. We simply need to evaluate $\delta\Gamma$ when A_i or F_{ij} is varied, and set the variation equal to zero, since this will give the effective field equation. Variation of (A12) leads to

$$\delta S_{\rm em} = \beta \int_\Sigma d\sigma_x \left\{ \nabla_j F^{ij} - J^i_{\rm ext} \right\} \delta A_i \ . \tag{A13}$$

Here ∇_i is the covariant derivative computed using the metric on Σ. We will define the ground state contribution to the current density by

$$\delta \tilde{S} = -\beta \int_\Sigma d\sigma_x \, J^i_{\rm ground} \delta A_i \ , \tag{A14}$$

as in Ref. [42]. We now need to compute $\delta\tilde{\Gamma}$. In the actual calculation of $\tilde{\Gamma}$, the result is expressed in terms of F_{ij} rather than directly in terms of A_i. However, it is easy to prove that

$$\frac{\delta\tilde{\Gamma}}{\delta A_i} = 2\nabla_j \left(\frac{\delta\tilde{\Gamma}}{\delta F_{ij}} \right) \ . \tag{A15}$$

If we define the current induced by quantum effects $J^i_{\rm ind}$ by [42]

$$J^i_{\rm ind} = -\frac{2}{\beta} \nabla_j \left(\frac{\delta\tilde{\Gamma}}{\delta F_{ij}} \right) \ , \tag{A16}$$

then the effective Maxwell equation obtained from $\dfrac{\delta\Gamma}{\delta A_i} = 0$ is

$$\nabla_j F^{ij} = J^i_{\text{ext}} + J^i_{\text{ground}} + J^i_{\text{ind}} \; . \tag{A17}$$

It is possible to rewrite (A17) in a form which resembles the result in the case of $D = 3$. This is done by defining the tensor H_{ij} by

$$H^{ij} = F^{ij} + 2T \frac{\delta \tilde{\Gamma}}{\delta F_{ij}} \; . \tag{A18}$$

As we will show in a moment, this is the analogue of the vector $\mathbf{H}$ in the usual $D = 3$ case. With the definition in (A18), we can rewrite (A17) as

$$\nabla_j H^{ij} = J^i_{\text{ext}} + J^i_{\text{ground}} \; . \tag{A19}$$

In order to see that our definitions correspond to the usual ones for $D = 3$, described in the earlier part of this appendix, restrict attention to $D = 3$. The magnetic field vector $\mathbf{B}$ with components B^i is defined in terms of the field strength tensor F_{ij} by $F_{ij} = \epsilon_{ijk} B^k$, or equivalently by $B^i = \frac{1}{2} \epsilon^{ijk} F_{jk}$, where ϵ_{ijk} is the Levi-Civitta tensor. In the same way we can define the vector $\mathbf{H}$ with components H^i in terms of the tensor H_{ij} by $H_{ij} = \epsilon_{ijk} H^k$ or $H^i = \frac{1}{2} \epsilon^{ijk} H_{jk}$. Contraction of both sides of (A18) with ϵ_{ijk}, and using

$$\frac{\delta \tilde{\Gamma}}{\delta F_{ij}} = \frac{1}{2} \epsilon^{ijk} \frac{\delta \tilde{\Gamma}}{\delta B^k} \; , \tag{A20}$$

results in

$$H_k = B_k + T \frac{\delta \tilde{\Gamma}}{\delta B^k} \; . \tag{A21}$$

The thermodynamic potential Ω is given by

$$\tilde{\Gamma} = \beta \Omega \; . \tag{A22}$$

If we define the thermodynamic potential density ω by

$$\Omega = \int_\Sigma d\sigma_x \, \omega \; , \tag{A23}$$

then we have

$$H_i = B_i - M_i \ , \tag{A24}$$

where

$$M_i = -\frac{\partial \omega}{\partial B^i} \tag{A25}$$

defines the components of the magnetization.

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